TWO-PARAMETER STOCHASTIC CALCULUS AND MALLIAVIN'S INTEGRATION-BY-PARTS FORMULA ON WIENER SPACE

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Abstract. The integration-by-parts formula discovered by Malliavin for the Itô map on Wiener space is proved using the two-parameter stochastic calculus. It is also shown that the solution of a one-parameter stochastic differential equation driven by a two-parameter semimartingale is itself a two-parameter semimartingale.

1. Introduction

The stochastic calculus of variations was conceived by Malliavin [6, 7, 8] as follows. Let $(z_t)_{t\geqslant 0}$ denote the Ornstein–Uhlenbeck process on Wiener space $(W, \mathcal{W}, \mu)$ and let $\Phi : W \to \mathbb{R}^d$ denote the (almost-everywhere unique) Itô map obtained by solving a stochastic differential equation in $\mathbb{R}^d$ up to time 1. Then $(z_t)_{t\geqslant 0}$ is stationary and reversible, so, for functions f, g on $\mathbb{R}^d$, setting $F = f \circ \Phi, G = g \circ \Phi$,

$$\mathbb{E}\left[\{F(z_t) - F(z_0)\}\{G(z_t) - G(z_0)\}\right] = -2\mathbb{E}\left[F(z_0)\{G(z_t) - G(z_0)\}\right]. \tag{1}$$

Once certain terms of mean zero are subtracted, a differentiation of this identity with respect to t *inside the expectation* is possible, and leads to the integration-by-parts formula on Wiener space

$$\int_W \nabla_i f(\Phi)\Gamma^{ij}\nabla_j g(\Phi) d\mu = -\int_W f(\Phi) LG d\mu, \tag{2}$$

where LG and the *covariance matrix* Γ will be defined below. As is now well known, this formula and its generalizations hold the key to many deep results of stochastic analysis.

Malliavin's proof of the integration-by-parts formula was based on a *transfer principle*, allowing some calculations for two-parameter random processes to be made using classical differential calculus. Stroock [11, 12, 13] and Shigekawa [10] gave alternative derivations having a a more functional-analytic flavour. Bismut [1] gave another derivation based on the Cameron–Martin–Girsanov formula. Elliott and Kohlmann [3] and Elworthy and Li [4] found further elementary approaches to the formula. The alternative proofs are relatively straightforward. Nevertheless, we have found it interesting to go back to Malliavin's original approach in [8] and to review the calculations needed, especially since this can be done now in a more explicit way using the two-parameter stochastic calculus, as formulated in [9].

In Section 2 we review in greater detail the various mathematical objects mentioned above. Then, in Section 3, we review some points of two-parameter stochastic calculus from [9]. Section 4 contains the main technical result of the paper, which is a regularity property for two-parameter stochastic differential equations. We consider equations in which some components are given by two-parameter integrals and others by one-parameter integrals. It is shown, under suitable hypotheses, that the components which are presented as one-parameter integrals are in fact two-parameter semimartingales. This is useful because

one can then compute martingale properties for both parameters by stochastic calculus. The sorts of differential equation to which this theory applies are just one way to realise continuous random processes indexed by the plane. See the survey [5] by Léandre for a wider discussion. But this regularity property makes our processes more tractable to analyse than some others. This is illustrated in Section 5, where we do the calculations needed to obtain the integration-by-parts formula.

2. Integration-by-parts formula

The Wiener space $(W, \mathcal{W}, \mu)$ over $\mathbb{R}^m$ is a probability space with underlying set $W = C([0,\infty), \mathbb{R}^m)$, the set of continuous paths in $\mathbb{R}^m$. Let $\mathcal{W}^o$ denote the σ-algebra on W generated by the family of coordinate functions $w \mapsto w_s : W \to \mathbb{R}^m$, $s \geqslant 0$, and let μ^o be Wiener measure on $\mathcal{W}^o$, that is to say, the law of a Brownian motion in $\mathbb{R}^m$ starting from 0. Then $(W, \mathcal{W}, \mu)$ is the completion of the probability space $(W, \mathcal{W}^o, \mu^o)$. Write $\mathcal{W}_s$ for the μ-completion of $\sigma(w \mapsto w_r : r \leqslant s)$. Let $X_0, X_1, \dots, X_m$ be vector fields on $\mathbb{R}^d$, with bounded derivatives of all orders. Fix $x_0 \in \mathbb{R}^d$ and consider the stochastic differential equation

$$\partial x_s = X_i(x_s)\partial w_s^i + X_0(x_s)\partial s.$$

Here and below, the index i is summed from 1 to m, and ∂ denotes the Stratonovich differential. There exists a map $x : [0,\infty) \times W \to \mathbb{R}^d$ with the following properties:

- x is a continuous semimartingale on $(W, \mathcal{W}, (\mathcal{W}_s)_{s \geqslant 0}, \mu)$,
- for μ-almost all $w \in W$, for all $s \geqslant 0$ we have

$$x_s(w) = x_0 + \int_0^s X_i(x_r(w))\partial w_r^i + \int_0^s X_0(x_r(w))dr.$$

The first integral in this equation is the Stratonovich stochastic integral. Moreover, for any other such map x', we have $x_s(w) = x'_s(w)$ for all $s \geqslant 0$, for μ-almost all w. We have chosen here a Stratonovich rather than an Itô formulation to be consistent with later sections, where we have made this choice in order to take advantage of the simpler calculations which the Stratonovich calculus allows. The Itô map referred to above is the map $\Phi(w) = x_1(w)$.

We can define on some complete probability space, $(\Omega, \mathcal{F}, \mathbb{P})$ say, a two-parameter, continuous, zero-mean Gaussian field $(z_{st} : s,t \geqslant 0)$ with values in $\mathbb{R}^m$, and with covariances given by

$$\mathbb{E}(z_{st}^i z_{s't'}^j) = \delta^{ij}(s \wedge s')e^{-|t-t'|/2}.$$

Such a field is called an Ornstein–Uhlenbeck sheet. Set $z_t = (z_{st} : s \geqslant 0)$. Then, for $t > 0$, both z_0 and z_t are Brownian motions in $\mathbb{R}^m$ and (z_0, z_t) and (z_t, z_0) have the same distribution. We have now defined all the terms in, and have justified, the identity (1).

Consider the following stochastic differential equation for an unknown process $(U_s : s \geqslant 0)$ in the space of $d \times d$ matrices

$$\partial U_s = \nabla X_i(x_s)U_s\partial w_s^i + \nabla X_0(x_s)U_s\partial s, \quad U_0 = I.$$

This equation may be solved, jointly with the equation for x, in exactly the same sense as the equation for x alone. Thus we obtain a map $U : [0,\infty) \times W \to \mathbb{R}^d \otimes (\mathbb{R}^d)^*$, with properties analogous to those of x. Moreover, by solving an equation for the inverse, we can

see that $U_s(w)$ remains invertible for all $s \geqslant 0$, for almost all w. Write U_s^* for the transpose matrix and set $\Gamma_s = U_s C_s U_s^*$, where

$$C_s = \int_0^s U_r^{-1} X_i(x_r) \otimes U_r^{-1} X_i(x_r) dr.$$

Set also

$$L_s = -U_s \int_0^s U_r^{-1} X_i(x_r) \partial w_r^i + U_s \int_0^s U_r^{-1} \{\nabla^2 X_i(x_r) \partial w_r^i + \nabla^2 X_0(x_r) dr\} \Gamma_r, \\ + U_s \int_0^s U_r^{-1} \nabla X_i(x_r) X_i(x_r) dr$$

and define for $G = g \circ \Phi$

$$LG = L_1^i \nabla_i g(x_1) + \Gamma_1^{ij} \nabla_i \nabla_j g(x_1).$$

We have now defined all the terms appearing in the integration-by-parts formula (2). We will give a proof in Section 5.

3. Review of two-parameter stochastic calculus

In [9], building on the fundamental works of Cairoli and Walsh [2] and Wong and Zakai [14, 15], we gave an account of two-parameter stochastic calculus, suitable for the development of a general theory of two-parameter hyperbolic stochastic differential equations. We recall here, for the reader's convenience, the main features of this account.

We take as our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the canonical complete probability space of an m-dimensional Brownian sheet $(w_{st} : s, t \geqslant 0)$, extended to a process $(w_{st} : s, t \in \mathbb{R})$ by independent copies in the other three quadrants. Thus $w_{st} = (w_{st}^1, \ldots, w_{st}^m)$ is a continuous, zero-mean Gaussian process, with covariances given by

$$\mathbb{E}(w_{st}^i w_{s't'}^j) = \delta^{ij}(s \wedge s')(t \wedge t'), \quad i, j = 1, \ldots, m, \quad s, t \geqslant 0, \quad s', t' \geqslant 0.$$

It will be convenient to define also $w_{st}^0 = st$ for all $s, t \in \mathbb{R}$. For $s, t \geqslant 0$, write $\mathcal{F}_{st}$ for the completion with respect to $\mathbb{P}$ of the σ-algebra generated by w_{ru} for $r \in (-\infty, s]$ and $u \in (-\infty, t]$. We say that a two-parameter process $(x_{st} : s, t \geqslant 0)$ is *adapted* if x_{st} is $\mathcal{F}_{st}$-measurable for all $s, t \geqslant 0$, and is *continuous* if $(s, t) \mapsto x_{st}(\omega)$ is continuous on $(\mathbb{R}^+)^2$ for all $\omega \in \Omega$. The *previsible* σ-algebra on $\Omega \times (\mathbb{R}^+)^2$ is that generated by sets of the form $A \times (s, s'] \times (t, t']$ with $A \in \mathcal{F}_{st}$. If we allow $A \in \mathcal{F}_{s\infty}$ in this definition, we get the *s-previsible* σ-algebra.

The classical approach to defining stochastic integrals, by means of an isometry of Hilbert spaces, adapts in a straightforward way from one-dimensional times to two, allowing the construction of stochastic integrals with respect to certain two-parameter processes, in particular with respect to the Brownian sheet. Given an s-previsible process[1] $(a_s(t) : s, t \geqslant 0)$, such that

$$\mathbb{E} \int_0^s \int_0^t a_r(u)^2 dr du < \infty$$

[1]We write any time parameter with respect to which a process is previsible, here s, as a subscript. Where previsibility is not assumed, here in t, we write the parameter in parentheses.

for all $s, t \geqslant 0$, we can define, for $i = 1, \dots, m$ and all $t_1, t_2 \geqslant 0$ with $t_1 \leqslant t_2$, one-parameter processes M and A by

$$M_s = \int_0^s \int_{t_1}^{t_2} a_r(t) d_r d_t w^i_{rt}, \quad A_s = \int_0^s \int_{t_1}^{t_2} a_r(t)^2 dr dt. \tag{3}$$

Then M is a continuous $(\mathcal{F}_{s\infty})_{s\geqslant 0}$-martingale, with quadratic variation process $[M] = A$. A localization argument by adapted initial open sets (see below) allows an extension of the integral under weaker integrability conditions. By the Burkholder–Davis–Gundy inequalities, for all $\alpha \in [2, \infty)$, there is a constant $C(\alpha) < \infty$ such that

$$\mathbb{E}\left(\left| \int_{s_1}^{s_2} \int_{t_1}^{t_2} a_s(t) d_s d_t w^i_{st} \right|^\alpha \right) \leqslant C(\alpha) \mathbb{E}\left(\left| \int_{s_1}^{s_2} \int_{t_1}^{t_2} a_s(t)^2 ds dt \right|^{\alpha/2} \right). \tag{4}$$

By an (s,t)-*semimartingale*, s-*semimartingale*, t-*semimartingale*, we mean, respectively, previsible processes $(x_{st} : s, t \geqslant 0)$, $(p_{st} : s, t \geqslant 0)$, $(q_{st} : s, t \geqslant 0)$ for which we may write

$$\begin{aligned} &x_{st} - x_{s0} - x_{0t} + x_{00} \\ &\quad = \sum_{i=0}^m \int_0^s \int_0^t (x''_{ru})_i d_r d_u w^i_{ru} + \sum_{i,j=0}^m \int_0^s \int_{-1}^t \left(\int_{-1}^s \int_0^t (x''_{ru}(r', u'))_{ij} d_{r'} d_u w^j_{r'u} \right) d_r d_{u'} w^i_{ru'} \end{aligned}$$

and

$$p_{st} - p_{0t} = \sum_{i=0}^m \int_0^s \int_{-1}^t (p'_{rt}(u'))_i d_r d_{u'} w^i_{ru'}, \quad q_{st} - q_{s0} = \sum_{i=0}^m \int_{-1}^s \int_0^t (q'_{su}(r'))_i d_{r'} d_u w^i_{r'u}.$$

Here, $(x''_{st} : s, t \geqslant 0)$ is a previsible process, having components $(x''_{st})_i$, subject to certain local integrability conditions, which are implied, in particular, by almost sure local boundedness. The process $(x''_{st}(r, u) : s, t \geqslant 0, r, u \in \mathbb{R})$ is required to be previsible in (ω, s, t) and (Borel) measurable in (r, u), with $x''_{st}(r, u) = 0$ for $r > s$ or $u > t$, and is subject to similar local integrability conditions. The inner and outer parts of the second integral are both cases of the stochastic integral at (3), or its t-analogue, or of the usual Lebesgue integral, and the value of the iterated integral is unchanged if we reverse the order in which the integrals are taken. The integrals appearing in the expression for x_{st} are called stochastic integrals of the *first* and *second kind*. The processes $(p'_{st}(u) : s, t \geqslant 0, u \in \mathbb{R})$ and $(q'_{st}(r) : s, t \geqslant 0, r \in \mathbb{R})$ are required to be previsible in (ω, s, t) and measurable in u and r, respectively, with $p'_{st}(u) = 0$ for $u > t$ and $q'_{st}(r) = 0$ for $r > s$, and are subject to similar local integrability conditions. For fixed $t \geqslant 0$, if $(x_{s0} : s \geqslant 0)$ is a continuous $(\mathcal{F}_{s0})_{s\geqslant 0}$-semimartingale, then $(x_{st} : s \geqslant 0)$ is a continuous $(\mathcal{F}_{st})_{s\geqslant 0}$-semimartingale, in the usual one-parameter sense. Also $(p_{st} : s \geqslant 0)$ is a continuous $(\mathcal{F}_{st})_{s\geqslant 0}$-semimartingale, for all $t \geqslant 0$.

The heuristic formulae

$$\begin{aligned}
d_s d_t x_{st} &= \sum_{i=0}^{m} (x''_{st})_i d_s d_t w^i_{st} + \sum_{i,j=0}^{m} \int_{-1}^{s} \int_{-1}^{t} (x''_{st}(r,u))_{ij} d_s d_u w^i_{su} d_r d_t w^j_{rt}, \\
d_s p_{st} &= \sum_{i=0}^{m} \int_{-1}^{t} (p'_{st}(u))_i d_s d_u w^i_{su}, \\
d_t q_{st} &= \sum_{i=0}^{m} \int_{-1}^{s} (q'_{st}(r))_i d_r d_t w^i_{rt}
\end{aligned}$$

provide a good intuition in representing the two-parameter increment

$$d_s d_t x_{st} = x_{s+ds,t+dt} - x_{s,t+dt} - x_{s+ds,t} + x_{st}$$

and the one-parameter increments $d_s p_{st} = p_{s+ds,t} - p_{st}$ and $d_t q_{st} = q_{s,t+dt} - q_{st}$ in terms of a linear combinations of increments, and of products of increments of the Brownian sheet.

By a (*two-parameter*) *semimartingale*, we mean a process which is at the same time an (s,t)-semimartingale, an s-semimartingale and a t-semimartingale. Such processes are necessarily continuous. An (s,t)-semimartingale which is constant on the s-axis and t-axis is a semimartingale. By an obvious choice of integrands, the process $(w_{st} : s,t \geqslant 0)$ is itself a semimartingale. The choice of lower limit -1 is useful to us in allowing as semimartingales a pair of independent $\mathbb{R}^m$-valued Brownian motions $(z_{s0} : s \geqslant 0)$ and $(b_{0t} : t \geqslant 0)$, given by

$$z_{s0} = \int_0^s \int_{-1}^0 d_r d_u w_{ru}, \quad b_{0t} = \int_{-1}^0 \int_0^t d_r d_u w_{ru},$$

which are moreover independent of $(w_{st} : s,t \geqslant 0)$. Here and below, we bring one-parameter processes defined on the s or t axes into the class of two-parameter processes by extending them as constant in the second parameter.

We say that a subset $\mathcal{D} \subseteq (\mathbb{R}^+)^2$ is an *initial open set* if it is non-empty and is a union of rectangles of the form $[0,s) \times [0,t)$, where $s,t \geqslant 0$. A random subset $\mathcal{D} \subseteq \Omega \times (\mathbb{R}^+)^2$ is *adapted* if the event $\{(s,t) \in \mathcal{D}\}$ is $\mathcal{F}_{st}$-measurable for all $s,t \geqslant 0$. For an adapted initial open set $\mathcal{D}$, a process $(x_{st} : (s,t) \in \mathcal{D})$ is a *semimartingale in* $\mathcal{D}$ if there exists a sequence of adapted initial open sets $\mathcal{D}_n \uparrow \mathcal{D}$, almost surely, and a sequence of semimartingales $(x^n_{st} : s,t \geqslant 0)$, such that $x_{st} = x^n_{st}$ for all $(s,t) \in \mathcal{D}_n$ for all n. The notion of an *s-semimartingale in* $\mathcal{D}$ is defined analogously. We write $\zeta(\mathcal{D})$ for the boundary of $\mathcal{D}$ as a subset of $(\mathbb{R}^+)^2$. In particular, if $\mathcal{D} = (\mathbb{R}^+)^2$, then $\zeta(\mathcal{D}) = \emptyset$.

The theory which we now describe is symmetrical in s and t. Where a statement is made for s, there is also a corresponding statement for t, which we shall often omit. Let $(x_{st} : s,t \geqslant 0)$ and $(x'_{st} : s,t \geqslant 0)$ be s-semimartingales and let $(a_{st} : s,t \geqslant 0)$ be a locally bounded previsible process, for example, a continuous adapted process. There exist s-semimartingales which, for each $t \geqslant 0$, provide versions of the one-parameter stochastic integral and the one-parameter covariation process

$$\zeta^1_{st} = \int_0^s a_{rt} d_r x_{rt}, \quad \zeta^2_{st} = \int_0^s d_r x_{rt} d_r x'_{rt}.$$

From now on, when we write these integrals, we assume that such a version has been chosen. We define also four types of two-parameter integral, each of which is a (two-parameter)

semimartingale. These are written

$$\zeta^3_{st} = \int_0^s \int_0^t a_{ru} d_r d_u x_{ru}, \quad \zeta^4_{st} = \int_0^s \int_0^t d_r x_{ru} d_u y_{ru},$$

$$\zeta^5_{st} = \int_0^s \int_0^t d_r x_{ru} d_r d_u y_{ru}, \quad \zeta^6_{st} = \int_0^s \int_0^t d_r d_u x_{ru} d_r d_u y_{ru}.$$

In the first and last integral, we require x to be an (s,t)-semimartingale, whereas, in the second and third, x should be an s-semimartingale. We require that y be a t-semimartingale in the second integral and an (s,t)-semimartingale in the third and fourth. All these integrals are defined as sums of certain integrals of the first and second kind with respect to the Brownian sheet. We refer to [9] for the details. We use the following differential notations:

$$\begin{array}{rcr}
d_s z_{st} = a_{st} d_s x_{st} & \text{means} & z_{st} - z_{0t} = \zeta^1_{st}, \\
d_s z_{st} = d_s x_{st} d_s x'_{st} & \text{means} & z_{st} - z_{0t} = \zeta^2_{st}, \\
d_s d_t z_{st} = a_{st} d_s d_t x_{st} & \text{means} & z_{st} - z_{s0} - z_{0t} + z_{00} = \zeta^3_{st}, \\
d_s d_t z_{st} = d_s x_{st} d_t y_{st} & \text{means} & z_{st} - z_{s0} - z_{0t} + z_{00} = \zeta^4_{st}, \\
d_s d_t z_{st} = d_s x_{st} d_s d_t y_{st} & \text{means} & z_{st} - z_{s0} - z_{0t} + z_{00} = \zeta^5_{st}, \\
d_s d_t z_{st} = d_s d_t x_{st} d_s d_t y_{st} & \text{means} & z_{st} - z_{s0} - z_{0t} + z_{00} = \zeta^6_{st}.
\end{array}$$

The integrals ζ^2_{st}, ζ^5_{st} and ζ^6_{st} all vanish if $d_s x_{st} = a_{st} ds$. It is shown in [9] that a series of identities hold among the various types of integral, which can be expressed conveniently in terms of this differential notation. Some identities assert the associativity of products involving a combination of three differentials or processes, the others are written as the following three rules

$$\begin{aligned}
d_s(f(x_{st})) &= f'(x_{st}) d_s x_{st} + \tfrac{1}{2} f''(x_{st}) d_s x_{st} d_s x_{st}, \\
d_s(a_{st} d_t x_{st}) &= d_s a_{st} d_t x_{st} + a_{st} d_s d_t x_{st} + d_s a_{st} d_s d_t x_{st}, \\
d_s(d_t x_{st} d_t y_{st}) &= d_s d_t x_{st} d_t y_{st} + d_t x_{st} d_s d_t y_{st} + d_s d_t x_{st} d_s d_t y_{st}.
\end{aligned}$$

These rules combine the usual calculus of partial differentials with Itô calculus in an obvious way. As a consequence, we can obtain a geometrically simpler Stratonovich-type calculus by defining, for processes $(x_{st} : s,t \geqslant 0)$ and $(y_{st} : s,t \geqslant 0)$, some further integrals, corresponding to the following differential rules

$$X_{st} \partial_s X_{st} = X_{st} dY_{st} + \tfrac{1}{2} d_s X_{st} d_s Y_{st}, \quad \partial_s X_{st} \partial_s Y_{st} = \partial_s X_{st} d_s Y_{st} = d_s X_{st} d_s Y_{st},$$

where X_{st} may stand for any one of $x_{st}, d_t x_{st}, \partial_t x_{st}$ and Y_{st} may stand for any one of $y_{st}, d_t y_{st}, \partial_t y_{st}$. Then we have

$$\begin{aligned}
\partial_s(f(x_{st})) &= f'(x_{st}) \partial_s x_{st}, \\
\partial_s(a_{st} \partial_t x_{st}) &= \partial_s a_{st} \partial_t x_{st} + a_{st} \partial_s \partial_t x_{st}, \\
\partial_s(\partial_t x_{st} \partial_t y_{st}) &= \partial_s \partial_t x_{st} \partial_t y_{st} + \partial_t x_{st} \partial_s \partial_t y_{st}.
\end{aligned}$$

The Brownian sheet $(w_{st} : s,t \geqslant 0)$ and the boundary Brownian motions $(z_{s0} : s \geqslant 0)$ and $(b_{0t} : t \geqslant 0)$ have some special properties, which are reflected in the following differential

formulae, for $1 \leqslant i, j \leqslant m$,

$$d_s d_t w^i_{st} d_s d_t w^j_{st} = \delta^{ij} ds dt, \quad d_s z^i_{s0} d_s z^j_{s0} = \delta^{ij} ds, \quad d_t b^i_{0t} d_t b^j_{0t} = \delta^{ij} dt,$$

and, for any semimartingale $(x_{st} : s, t \geqslant 0)$,

$$d_s x_{st} d_s d_t w^i_{st} = d_t x_{st} d_s d_t w^i_{st} = 0.$$

4. A regularity result for two-parameter stochastic differential equations

We discussed in [9] a class of two-parameter hyperbolic stochastic differential equations, in which there is given, for a system of processes $(x_{st}, p_{st}, q_{st} : s, t \geqslant 0)$, one equation for the mixed second-order differential $d_s d_t x_{st}$, together with two further equations for the one-parameter differentials $d_s p_{st}$ and $d_t q_{st}$. We review briefly the details below, and then give a new regularity result, which we need for our application to Malliavin's integration-by-parts formula, but which may be of independent interest. This result concerns the process $(p_{st} : s, t \geqslant 0)$ (and analogously also $(q_{st} : s, t \geqslant 0)$), which, since integrated in s, has naturally the regularity of an s-semimartingale. The point at issue is whether $(p_{st} : s, t \geqslant 0)$ is a full (two-parameter) semimartingale. A method to establish this is stated in [9, pp. 299, 315-316], but the argument given is incomplete. A full proof is given below in Theorem 4.2. As an illustrative example, we note that, if $(w_{st} : s, t \geqslant 0)$ is a Brownian sheet with values in $\mathbb{R}^m$, then the result will show that there is a two-parameter semimartingale $(x_{st} : s, t \geqslant 0)$ such that, for all $t \geqslant 0$, the process $(x_{st} : s \geqslant 0)$ satisfies the one-parameter stochastic differential equation

$$\partial_s x_{st} = X_i(x_{st}) \partial_s w^i_{st} + X_0(x_{st}) \partial s,$$

with given initial values $x_{0t} = x_0$, say. This is useful because, now, despite the irregular dependence of the Brownian sheet on t, we can use a differential calculus in t as well as in s.

Consider the class of hyperbolic stochastic differential equations in $(\mathbb{R}^+)^2$ of the form

$$d_s d_t x_{st} = a(d_s d_t w_{st}) + b(d_s x_{st}, d_t x_{st}), \tag{5}$$

$$d_s p_{st} = c(d_s x_{st}), \tag{6}$$

$$d_t q_{st} = e(d_t x_{st}). \tag{7}$$

Here $w_{st} = (w^1_{st}, \dots, w^m_{st})$,with $(w^i_{st} : s, t \geqslant 0)$, $i = 1, \dots, m$, independent Brownian sheets, as above. The unknown processes $(x_{st} : s, t \geqslant 0)$, $(p_{st} : s, t \geqslant 0)$ and $(q_{st} : s, t \geqslant 0)$ take values in $\mathbb{R}^d$, $\mathbb{R}^n$ and $\mathbb{R}^n$, respectively, and are subject to given boundary values $(x_{s0} : s \geqslant 0)$, $(x_{0t} : t \geqslant 0)$, both assumed to be semimartingales, and $(p_{0t} : t \geqslant 0)$, $(q_{s0} : s \geqslant 0)$, both assumed continuous and adapted. The coefficients a, b, c, e are allowed to have a locally Lipschitz dependence on the unknown processes, with the restriction that b depends only on x. Thus, for example, we would write $a(x_{st}, p_{st}, q_{st}, d_s d_t w_{st})$ and $b(x_{st}, d_s x_{st}, d_t x_{st})$, but have not done so in order to keep the notation compact. Moreover, we allow a dependence on the differentials which is a sum of linear and quadratic terms. Thus, in an expanded

notation, we would write

$$\begin{aligned} d_s d_t x_{st} &= a_1(d_s d_t w_{st}) + a_2(d_s d_t w_{st}, d_s d_t w_{st}) \\ &\quad + b_{11}(d_s x_{st}, d_t x_{st}) + b_{12}(d_s x_{st}, d_t x_{st}, d_t x_{st}), \\ &\quad + b_{21}(d_s x_{st}, d_s x_{st}, d_t x_{st}) + b_{22}(d_s x_{st}, d_s x_{st}, d_t x_{st}, d_t x_{st}), \\ d_s p_{st} &= c_1(d_s x_{st}) + c_2(d_s x_{st}, d_s x_{st}), \\ d_t q_{st} &= e_1(d_t x_{st}) + e_2(d_t x_{st}, d_t x_{st}), \end{aligned}$$

where, for $i, j, k = 1, 2$,

$$\begin{aligned} a_i &: \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d \otimes ((\mathbb{R}^m)^*)^{\otimes i}, \\ b_{jk} &: \mathbb{R}^d \to \mathbb{R}^d \otimes ((\mathbb{R}^d)^*)^{\otimes j+k}, \\ c_j &: \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes ((\mathbb{R}^d)^*)^{\otimes j}, \\ e_k &: \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes ((\mathbb{R}^d)^*)^{\otimes k}. \end{aligned}$$

We may and do assume with loss that $a_2, b_{12}, b_{21}, b_{22}, c_2, e_2$ are symmetric in any pair of repeated differential arguments.

By a *local solution of* (5–7) *with domain* $\mathcal{D}$ we mean an adapted initial open set $\mathcal{D}$, together with a semimartingale $(x_{st} : (s,t) \in \mathcal{D})$, an s-semimartingale $(p_{st} : (s,t) \in \mathcal{D})$, and a t-semimartingale $(q_{st} : (s,t) \in \mathcal{D})$, all continuous on $\mathcal{D}$, such that, for all $(s,t) \in \mathcal{D}$,

$$\begin{aligned} x_{st} &= x_{s0} + x_{0t} - x_{00} + \int_0^s \int_0^t a(d_r d_u w_{ru}) + \int_0^s \int_0^t b(d_r x_{ru}, d_u x_{ru}), \\ p_{st} &= p_{0t} + \int_0^s c(d_r x_{rt}), \\ q_{st} &= q_{s0} + \int_0^t e(d_u x_{su}). \end{aligned}$$

Given such a solution, for each $t \geqslant 0$, we can define processes $(u_{st} : (s,t) \in \mathcal{D})$ and $(u^*_{st} : (s,t) \in \mathcal{D})$, taking values in $\mathbb{R}^d \times (\mathbb{R}^d)^*$ and $\mathbb{R}^d \times (\mathbb{R}^d)^* \times (\mathbb{R}^d)^*$ respectively, by solving the linear one-parameter stochastic differential equations

$$d_s u_{st} = b_{11}(d_s x_{st}, \cdot) u_{st} + b_{12}(d_s x_{st}, d_s x_{st}, \cdot) u_{st}, \tag{8}$$

$$\begin{aligned} d_s u^*_{st} &= u_{st}^{-1}\{b_{12}(d_s x_{st}, u_{st}\cdot, u_{st}\cdot) \\ &\quad + b_{22}(d_s x_{st}, d_s x_{st}, u_{st}\cdot, u_{st}\cdot) - b_{11}(d_s x_{st}, b_{12}(d_s x_{st}, u_{st}\cdot, u_{st}\cdot))\}. \end{aligned} \tag{9}$$

Here u_{st}^{-1} denotes the inverse of the linear map u_{st}. For fixed $t \geqslant 0$, almost surely, u_{st} remains in the set of invertible maps while $(s,t) \in \mathcal{D}$. To see this, one can obtain formally a linear equation for the process $(u_{st}^{-1} : (s,t) \in \mathcal{D})$, and then check that its solution is indeed an inverse for u_{st}. Similarly, for each $s \geqslant 0$, we can define processes $(v_{st} : (s,t) \in \mathcal{D})$ and $(v^*_{st} : (s,t) \in \mathcal{D})$, taking values in $\mathbb{R}^d \times (\mathbb{R}^d)^*$ and $\mathbb{R}^d \times (\mathbb{R}^d)^* \times (\mathbb{R}^d)^*$, by solving the analogous equations

$$d_t v_{st} = b_{11}(\cdot, d_t x_{st}) v_{st} + b_{21}(\cdot, d_t x_{st}, d_t x_{st}) v_{st}. \tag{10}$$

$$\begin{aligned} d_t v^*_{st} &= v_{st}^{-1}\{b_{21}(v_{st}\cdot, v_{st}\cdot, d_t x_{st}) \\ &\quad + b_{22}(v_{st}\cdot, v_{st}\cdot, d_t x_{st}, d_t x_{st}) - b_{11}(b_{21}(v_{st}\cdot, v_{st}\cdot, d_t x_{st}), d_t x_{st})\}. \end{aligned} \tag{11}$$

We specify initial conditions $u_{00} = v_{00} = I$, so determining completely $(u_{0s} : s \geqslant 0)$ and $(v_{0t} : t \geqslant 0)$. Then we complete the determination of the above processes by specifying that $u_{0t} = v_{0t}$, $u^*_{0t} = 0$, $v_{s0} = u_{s0}$, and $v^*_{s0} = 0$ for all $s, t \geqslant 0$. Let us say that $(x_{st}, p_{st}, q_{st} : (s,t) \in \mathcal{D})$ is a *regular* local solution[2] if there exist continuous s-semimartingales $(u_{st} : (s,t) \in \mathcal{D})$ and $(u^*_{st} : (s,t) \in \mathcal{D})$ satisfying, for each $t \geqslant 0$, the equations (8–9), and if there exist also continuous t-semimartingales $(v_{st} : (s,t) \in \mathcal{D})$ and $(v^*_{st} : (s,t) \in \mathcal{D})$ satisfying, for each $s \geqslant 0$, the equations (10–11). A local solution is *maximal* if it is not the restriction of any local solution with larger domain. The notion of a maximal regular local solution is defined analogously. We assume that the boundary semimartingales $(x_{s0} : s \geqslant 0)$, $(x_{0t} : t \geqslant 0)$, $(p_{0t} : t \geqslant 0)$ and $(q_{s0} : s \geqslant 0)$ are *regular*[3]. By this we mean that the Lebesgue–Stieltjes measures defined by their quadratic variation processes and by the total variation processes of their finite variation parts are all dominated by Kds, or Kdt as appropriate, for some constant $K < \infty$. We give a result first for the case where $b = 0$.

Lemma 4.1. *Assume that $b = 0$. Let U be an open subset of $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n$ and let $m : U \to [0, \infty)$ be a continuous function with $m(x,p,q) \to \infty$ as $(x,p,q) \to \partial U$. Assume that, for all $M \geqslant 1$, the coefficients a, c, e are bounded and Lipschitz on the set $U_M = \{(x,p,q) \in U : m(x,p,q) < M\}$. Then, for any set of regular boundary semimartingales $(x_{s0} : s \geqslant 0)$, $(x_{0t} : t \geqslant 0)$, $(p_{0t} : t \geqslant 0)$ and $(q_{s0} : s \geqslant 0)$, with $(x_{00}, p_{00}, q_{00}) \in U$, the equations (5–7) have a unique maximal local solution $(x_{st}, p_{st}, q_{st} : (s,t) \in \mathcal{D})$ with values in U. Moreover, we have, almost surely*[4]

$$\sup_{r \leqslant s, u \leqslant t} m(x_{ru}, p_{ru}, q_{ru}) \to \infty \quad \text{as} \quad (s,t) \uparrow \zeta(\mathcal{D}).$$

Proof. In the case where m is bounded (so $U_M = U = \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n$ for large M), the existence of a (global) solution is proved in [9, Theorem 3.2.2]. The proof is of a standard type, using Picard iteration, Gronwall's lemma and Kolmogorov's continuity criterion, and gives also the uniqueness of local solutions on the intersections of their domains. When m is unbounded, we can find, for each $M \geqslant 1$, bounded Lipschitz coefficients a_M, c_M, e_M on $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n$, which agree with a, c, e on U_M. For each $M_0 \geqslant 1$, the corresponding global solutions $(x^M_{st}, p^M_{st}, q^M_{st} : s, t \geqslant 0)$ agree, for all integers $M \geqslant M_0$, almost surely, on $\mathcal{D}_{M_0}$, where

$$\mathcal{D}_M = \{(s,t) \in (\mathbb{R}^+)^2 : \sup_{r \leqslant s, u \leqslant t} m(x^M_{ru}, p^M_{ru}, q^M_{ru}) \leqslant M\}.$$

Hence, we obtain a local solution with all the claimed properties by setting $\mathcal{D} = \cup_M \mathcal{D}_M$ and by setting, for all $M \geqslant 1$, $(x_{st}, p_{st}, q_{st}) = (x^M_{st}, p^M_{st}, q^M_{st})$ for all $(s,t) \in \mathcal{D}_M \setminus \mathcal{D}_{M-1}$. $\square$

Our main result deals with the case when b is non-zero.

Theorem 4.2. *Assume that the coefficients a, b, c, e are uniformly bounded and Lipschitz. Then, for each set of regular semimartingale boundary values $(x_{s0} : s \geqslant 0)$, $(x_{0t} : t \geqslant 0)$,*

[2]It is not hard to see that, for any local solution, the processes just defined have previsible versions, which are then s-semimartingales or t-semimartingales, depending on the variable of integration. However, we have not determined whether they have a continuous version in general.

[3]No connection with the notion of regular local solution is intended.

[4]To clarify, we mean that, for all $(s^*, t^*) \in \zeta(\mathcal{D})$, the given limit holds whenever $(s,t) \uparrow (s^*, t^*)$. In particular, in the case where $\mathcal{D} = (\mathbb{R}^+)^2$, there are no such points (s^*, t^*) and nothing is claimed.

$(p_{0t} : t \geqslant 0)$, $(q_{s0} : s \geqslant 0)$, *the system of equations* (5–7) *has a unique maximal regular solution, with domain* $\mathcal{D}$ *say. As* $(s,t) \uparrow \zeta(\mathcal{D})$, *we have*

$$m_{st} = \sup_{s' \leqslant s, t' \leqslant t} |(u_{s't'}, u_{s't'}^{-1}, v_{s't'}, v_{s't'}^{-1})| \to \infty. \tag{12}$$

Moreover, if c *has Lipschitz first and second derivatives and has no dependence on* q, *then* $(p_{st} : s,t \in \mathcal{D})$ *is a semimartingale in* $\mathcal{D}$.

Proof. We consider first the question of existence. We follow, to begin, the strategy used in the proof of [9, Theorem 3.2.3]. Consider the following system of differential equations, for unknown processes $y_{st}, z_{st}, x'_{st}, u_{st}, u^*_{st}, p_{st}, x''_{st}, v_{st}, v^*_{st}, q_{st}$, taking values in $\mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^d \otimes (\mathbb{R}^d)^*, \mathbb{R}^d \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*, \mathbb{R}^n, \mathbb{R}^d, \mathbb{R}^d \otimes (\mathbb{R}^d)^*, \mathbb{R}^d \otimes (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^*, \mathbb{R}^n$ respectively:

$$d_s d_t y_{st} = u_{st}^{-1} a(d_s d_t w_{st}) - u^*_{st}(u_{st}^{-1} a(d_s d_t w_{st}) \otimes u_{st}^{-1} a(d_s d_t w_{st})), \tag{13}$$

$$d_s d_t z_{st} = v_{st}^{-1} a(d_s d_t w_{st}) - v^*_{st}(v_{st}^{-1} a(d_s d_t w_{st}) \otimes v_{st}^{-1} a(d_s d_t w_{st})), \tag{14}$$

$$d_s x'_{st} = v_{st}(d_s z_{st} + v^*_{st} d_s z_{st} \otimes d_s z_{st}), \tag{15}$$

$$d_s u_{st} = b_{11}(v_{st}(d_s z_{st} + v^*_{st} d_s z_{st} \otimes d_s z_{st}), \cdot) u_{st} + b_{21}(v_{st} d_s z_{st}, v_{st} d_s z_{st}, \cdot) u_{st}, \tag{16}$$

$$\begin{aligned} d_s u^*_{st} = u_{st}^{-1}\{ & b_{12}(v_{st}(d_s z_{st} + v^*_{st} d_s z_{st} \otimes d_s z_{st}), u_{st}\cdot, u_{st}\cdot) \\ & + b_{22}(v_{st} d_s z_{st}, v_{st} d_s z_{st}, u_{st}\cdot, u_{st}\cdot) - b_{11}(v_{st} d_s z_{st}, b_{12}(v_{st} d_s z_{st}, u_{st}\cdot, u_{st}\cdot))\}, \end{aligned} \tag{17}$$

$$d_s p_{st} = c(v_{st}(d_s z_{st} + v^*_{st} d_s z_{st} \otimes d_s z_{st})), \tag{18}$$

$$d_t x''_{st} = u_{st}(d_t y_{st} + u^*_{st} d_t y_{st} \otimes d_t y_{st}), \tag{19}$$

$$d_t v_{st} = b_{11}(\cdot, u_{st}(d_t y_{st} + u^*_{st} d_t y_{st} \otimes d_t y_{st})) v_{st} + b_{12}(\cdot, u_{st} d_t y_{st}, u_{st} d_t y_{st}) v_{st}, \tag{20}$$

$$\begin{aligned} d_t v^*_{st} = v_{st}^{-1}\{ & b_{21}(v_{st}\cdot, v_{st}\cdot, u_{st}(d_t y_{st} + u^*_{st} d_t y_{st} \otimes d_t y_{st})) \\ & + b_{22}(v_{st}\cdot, v_{st}\cdot, u_{st} d_t y_{st}, u_{st} d_t y_{st}) - b_{11}(b_{21}(v_{st}\cdot, v_{st}\cdot, u_{st} d_t y_{st}), u_{st} d_t y_{st})\}, \end{aligned} \tag{21}$$

$$d_t q_{st} = e(u_{st}(d_t y_{st} + u^*_{st} d_t y_{st} \otimes d_t y_{st})). \tag{22}$$

We evaluate the coefficients a, b, c and e here at $(x'_{st}, p_{st}, q_{st})$ (rather than at x''_{st}). Note that this system has the same form as the system (5–7) with $b = 0$. We use the boundary conditions given above for $u_{st}, p_{st}, v_{st}, q_{st}$. Define boundary values for y_{st} and z_{st} by

$$d_s y_{s0} = d_s z_{s0} = v_{s0}^{-1} d_s x_{s0}, \quad d_t y_{0t} = d_t z_{0t} = u_{0t}^{-1} d_t x_{0t}, \quad y_{00} = z_{00} = 0. \tag{23}$$

Set $u^*_{0t} = v^*_{s0} = 0$ and use the given boundary values $(x_{0t} : t \geqslant 0)$ for x'_{st} and $(x_{s0} : s \geqslant 0)$ for x''_{st}. Define, on the set U where u and v are invertible,

$$m(y, z, x', u, u^*, p, x'', v, v^*, q) = |(u, u^{-1}, v, v^{-1})| + |(u^*, v^*)|.$$

Then the preceding lemma applies, to show that (13–22) has a unique maximal local solution with the given boundary values, with domain $\mathcal{D}$ say, such that u_{st} and v_{st} are invertible for all $(s,t) \in \mathcal{D}$, and such that, almost surely, as $t \uparrow \zeta(\mathcal{D})$, either

$$m_{st} = \sup_{s' \leqslant s, t' \leqslant t} |(u_{s't'}, u_{s't'}^{-1}, v_{s't'}, v_{s't'}^{-1})| \uparrow \infty, \tag{24}$$

or

$$n_{st} = \sup_{s' \leqslant s, t' \leqslant t} |(u^*_{s't'}, v^*_{s't'})| \uparrow \infty. \tag{25}$$

Now v_{st} and v^*_{st} are continuous t-semimartingales (in $\mathcal{D}$) and z_{st} is a semimartingale. Moreover $d_t a_{st} d_s d_t z_{st} = 0$ for any t-semimartingale a_{st}. Hence, by [9, Theorem 2.3.1], x'_{st} is a semimartingale and we may take the t-differential in (15) to obtain

$$\begin{aligned} d_s d_t x'_{st} &= d_t v_{st}(d_s z_{st} + v^*_{st} d_s z_{st} \otimes d_s z_{st}) \\ &\quad + v_{st}(d_s d_t z_{st} + d_t v^*_{st} d_s z_{st} \otimes d_s z_{st} + v^*_{st} d_s d_t z_{st} \otimes d_s d_t z_{st}) + d_t v_{st}(d_t v^*_{st} d_s z_{st} \otimes d_s z_{st}) \\ &= a(d_s d_t w_{st}) + b(d_s x'_{st}, d_t x''_{st}). \end{aligned}$$

Similarly, by taking the s-differential in (19), we obtain

$$d_s d_t x''_{st} = a(d_s d_t w_{st}) + b(d_s x'_{st}, d_t x''_{st}).$$

We also have $x'_{00} = x''_{00}$ and

$$d_s x'_{s0} = v_{s0} d_s z_{s0} = d_s x''_{s0}, \quad d_t x'_{0t} = u_{0t} d_t y_{0t} = d_t x''_{0t},$$

so $x'_{st} = x''_{st}$ for all $(s,t) \in \mathcal{D}$, almost surely. Denote the common value of these processes by x_{st}. Then $(x_{st} : (s,t) \in \mathcal{D})$ satisfies (5). On using (15) and (19) to substitute[5] for $d_s z_{st}$ and $d_t y_{st}$ in (16, 18, 20, 22), we see also that p_{st}, q_{st}, u_{st}, u^*_{st}, v_{st}, v^*_{st} satisfy (6–11) respectively. Hence $(x_{st}, p_{st}, q_{st} : (s,t) \in \mathcal{D})$ is a regular local solution to (5–7), which is moreover maximal by virtue of (24–25).

We turn to the question of uniqueness. Suppose that $(\tilde{x}_{st}, \tilde{p}_{st}, \tilde{q}_{st} : (s,t) \in \tilde{\mathcal{D}})$ is any regular local solution to (5–7). Write $(\tilde{u}_{st}, \tilde{u}^*_{st}, \tilde{v}_{st}, \tilde{v}^*_{st} : (s,t) \in \tilde{\mathcal{D}})$ for the associated processes, satisfying (8–11). Define semimartingales $(\tilde{y}_{st} : (s,t) \in \tilde{\mathcal{D}})$ and $(\tilde{z}_{st} : (s,t) \in \tilde{\mathcal{D}})$ by

$$d_s d_t \tilde{y}_{st} = \tilde{u}^{-1}_{st} a(d_s d_t w_{st}) - \tilde{u}^*_{st}(\tilde{u}^{-1}_{st} a(d_s d_t w_{st}) \otimes \tilde{u}^{-1}_{st} a(d_s d_t w_{st})), \tag{26}$$

$$d_s d_t \tilde{z}_{st} = \tilde{v}^{-1}_{st} a(d_s d_t w_{st}) - \tilde{v}^*_{st}(\tilde{v}^{-1}_{st} a(d_s d_t w_{st}) \otimes \tilde{v}^{-1}_{st} a(d_s d_t w_{st})), \tag{27}$$

with boundary values (23). The following equations may be verified by checking that the initial values and differentials of left and right hand sides agree

$$d_s \tilde{x}_{st} = \tilde{v}_{st}(d_s \tilde{z}_{st} + \tilde{v}^*_{st} d_s \tilde{z}_{st} \otimes d_s \tilde{z}_{st}), \quad d_t \tilde{x}_{st} = \tilde{u}_{st}(d_t \tilde{y}_{st} + \tilde{u}^*_{st} d_t \tilde{y}_{st} \otimes d_t \tilde{y}_{st}). \tag{28}$$

Then, using these equations to substitute for $d_s \tilde{x}_{st}$ and $d_t \tilde{x}_{st}$ in (6–11), we see that $(\tilde{y}_{st}, \tilde{z}_{st}, \tilde{x}_{st}, \tilde{u}_{st}, \tilde{u}^*_{st}, \tilde{p}_{st}, \tilde{x}_{st}, \tilde{v}_{st}, \tilde{v}^*_{st}, \tilde{q}_{st} : (s,t) \in \tilde{\mathcal{D}})$ is a local solution to (13–22). By local uniqueness for this system, $\tilde{\mathcal{D}} \subseteq \mathcal{D}$ and $(\tilde{x}_{st}, \tilde{p}_{st}, \tilde{q}_{st}) = (x_{st}, p_{st}, q_{st})$ for all $(s,t) \in \tilde{\mathcal{D}}$, almost surely. Thus $(x_{st}, p_{st}, q_{st} : (s,t) \in \mathcal{D})$ is the unique maximal regular local solution to (5–7).

Our next goal is to obtain αth-moment and L^α-Hölder estimates on the process $(x_{st}, p_{st}, q_{st}, u_{st}, u^*_{st}, v_{st}, v^*_{st} : (s,t) \in \mathcal{D})$, for $\alpha \in [2,\infty)$. Write K for a uniform bound on a, b, c, e which is also a Lipschitz constant for b. Fix $M, N, T \geqslant 1$ and set

$$\mathcal{D}_M = \{(s,t) \in \mathcal{D} : s,t \leqslant T \text{ and } m_{st} \leqslant M\},$$
$$\mathcal{D}_{M,N} = \{(s,t) \in \mathcal{D} : s,t \leqslant T, m_{st} \leqslant M \text{ and } n_{st} \leqslant N\}.$$

Fix α and define

$$g(s,t) = \sup_{s' \leqslant s, t' \leqslant t} \mathbb{E}(|(u^*_{s't'}, v^*_{s't'})|^\alpha 1_{\{(s',t') \in \mathcal{D}_{M,N}\}}).$$

[5] Such substitutions result in differential formulae corresponding to valid identities between processes. This is because the two-parameter stochastic differential calculus is associative, as mentioned above, and as discussed in [9, pp. 290–291].

Let $(a_s : s \geqslant 0)$ be a locally bounded, $(\mathcal{F}_{s\infty})_{s\geqslant 0}$-previsible process. The following identities follow from equations (27) and (28): for $(s,t) \in \mathcal{D}$, respectively in $\mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d$,
(29)
$$\int_0^s a_r d_r x_{rt} = \int_0^s a_r d_r x_{r0} + \int_0^s \int_0^t a_r v_{rt} \left\{ v_{ru}^{-1} a(d_r d_u w_{ru}) + (v_{rt}^* - v_{ru}^*)(v_{ru}^{-1} a(d_r d_u w_{ru}))^{\otimes 2} \right\}$$
and
$$\int_0^s a_r d_r x_{rt} \otimes d_r x_{rt} = \int_0^s a_r d_r x_{r0} \otimes d_r x_{r0} + \int_0^s \int_0^t a_r (v_{rt} v_{ru}^{-1} a(d_r d_u w_{ru}))^{\otimes 2}. \tag{30}$$
Hence, using the estimate (4), we obtain a constant $C = C(\alpha, K, M, T) < \infty$ such that, for all $s, t \geqslant 0$,
$$\begin{aligned} &\mathbb{E}\left(\left| \int_0^s a_r d_r x_{rt} \right|^\alpha 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right) \\ &\qquad \leqslant C\mathbb{E}\left(\left| \left(\int_0^s a_r^2 dr \right)^{1/2} + \int_0^s \int_0^t |a_r|(|v_{rt}^*| + |v_{ru}^*|) dr du \right|^\alpha 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right) \end{aligned} \tag{31}$$
and
$$\mathbb{E}\left(\left| \int_0^s a_r d_r x_{rt} \otimes d_r x_{rt} \right|^\alpha 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right) \leqslant C\mathbb{E}\left(\left| \int_0^s |a_r| dr \right|^\alpha 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right). \tag{32}$$
Here and below, we suppress any dependence of constants on the dimensions d, n, m. If we allow C to depend also on N, then (31) may be simplified to
$$\mathbb{E}\left(\left| \int_0^s a_r d_r x_{rt} \right|^\alpha 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right) \leqslant C\mathbb{E}\left(\left| \int_0^s a_r^2 dr \right|^{\alpha/2} 1_{\{(s,t)\in\mathcal{D}_{M,N}\}} \right) \tag{33}$$
We use these estimates, along with analogous estimates for integrals $d_t x_{st}$, in the equations (9) and (11), to arrive at the inequality
$$g(s,t) \leqslant C \left(1 + \int_0^s g(s', t) ds' + \int_0^t g(s, t') dt' \right),$$
for a constant $C = C(\alpha, K, M, T) < \infty$. Since $N < \infty$, we know that $g(s,t) < \infty$ for all s, t, so this inequality implies that $g(s,t) \leqslant C$ for another constant $C < \infty$ of the same dependence. Similar arguments yield a further constant $C < \infty$ of the same dependence such that, for all $s, s' \geqslant 0$ and all $t, t' \geqslant 0$,
$$\mathbb{E}(|(x_{st}, u_{st}, u_{st}^*, p_{st}) - (x_{s't}, u_{s't}, u_{s't}^*, p_{s't})|^\alpha 1_{\{(s,t),(s',t)\in\mathcal{D}_{M,N}\}}) \leqslant C|s - s'|^{\alpha/2} \tag{34}$$
and
$$\mathbb{E}(|(x_{st}, v_{st}, v_{st}^*, q_{st}) - (x_{st'}, v_{st'}, v_{st'}^*, q_{st'})|^\alpha 1_{\{(s,t),(s,t')\in\mathcal{D}_{M,N}\}}) \leqslant C|t - t'|^{\alpha/2}. \tag{35}$$
Here, we have used Cauchy–Schwarz to obtain in an intermediate step
$$\int_s^{s'} \int_0^t |v_{ru}^*| dr du \leqslant |s - s'|^{1/2} \left(\int_s^{s'} \int_0^t |v_{ru}^*|^2 dr du \right)^{1/2}.$$

On going back to (29) and (30) with these Hölder estimates, we obtain, using (4) again, a constant $C < \infty$ of the same dependence such that

$$\mathbb{E}\left(\left|\int_0^s a_r(d_r x_{rt} - d_r x_{rt'})\right|^\alpha 1_{\{(s,t),(s,t')\in\mathcal{D}_{M,N}\}}\right) \leqslant C|t-t'|^{\alpha/2}\left(\mathbb{E}\left|\int_0^s a_r^2 ds\right|^\alpha\right)^{1/2} \tag{36}$$

and

$$\mathbb{E}\left(\left|\int_0^s a_r d_r x_{rt} \otimes (d_r x_{rt} - d_r x_{rt'})\right|^\alpha 1_{\{(s,t),(s,t')\in\mathcal{D}_{M,N}\}}\right) \leqslant C|t-t'|^{\alpha/2}\left(\mathbb{E}\left|\int_0^s a_r^2 ds\right|^\alpha\right)^{1/2}. \tag{37}$$

Now

$$\begin{aligned} d_s(u_{st}^{-1}u_{st'}) &= u_{st}^{-1}\{b(x_{st'}, d_s x_{st'}, \cdot) - b(x_{st}, d_s x_{st}, \cdot)\}u_{st'} \\ &\quad - u_{st}^{-1}b_{11}(x_{st}, d_s x_{st}, \cdot)\{b_{11}(x_{st'}, d_s x_{st'}, \cdot) - b_{11}(x_{st}, d_s x_{st}, \cdot)\}u_{st'}. \end{aligned}$$

We have made explicit the dependence of b and b_{11} on x_{st} or $x_{st'}$. We use the estimates (31), (32), (35–37) to find a constant $C = C(\alpha, K, M, T) < \infty$ such that

$$\mathbb{E}(|u_{st} - u_{st'}|^\alpha 1_{\{(s,t),(s,t')\in\mathcal{D}_{M,N}\}}) \leqslant C|t-t'|^{\alpha/2}. \tag{38}$$

Moreover, the same estimates, applied to the difference of (9) at t and at t', show that C may be chosen such that

$$\mathbb{E}(|u^*_{st} - u^*_{st'}|^\alpha 1_{\{(s,t),(s,t')\in\mathcal{D}_{M,N}\}}) \leqslant C|t-t'|^{\alpha/2}. \tag{39}$$

Since C does not depend on N, by monotone convergence, we can replace $\mathcal{D}_{M,N}$ by $\mathcal{D}_M$ in these estimates By symmetry, there are analogous estimates for v_{st} and v^*_{st}. Hence, using [9, Theorem 3.2.1], almost surely, for all $M \geqslant 1$, n_{st} remains bounded on $\mathcal{D}_M$. Thus (25) implies (24) so, in any case, (12) holds.

It remains to consider the case where c has Lipschitz first and second derivatives and has no dependence on q, and to show then that $(p_{st} : (s,t) \in \mathcal{D})$ is a semimartingale. For ease of writing, we shall assume that c has no dependence on x either. This is done without loss of generality, by the device of adding to our system the equation $d_s x_{st} = d_s x_{st}$, thus making x_{st} a component of p_{st}.

We seek to find a solution in a smaller class of processes, in which p_{st} is a semimartingale. Recall that

$$d_s p_{st} = c(d_s x_{st}) = c_1(p_{st})(d_s x_{st}) + c_2(p_{st})(d_s x_{st}, d_s x_{st}). \tag{40}$$

By Itô's formula, if p_{st} is a semimartingale, then

$$\begin{aligned} d_s d_t p_{st} &= c'(d_t p_{st}, d_s x_{st}) + \tfrac{1}{2}c''(d_t p_{st}, d_t p_{st}, d_s x_{st}) + c(d_s d_t x_{st}) + c'(d_t p_{st}, d_s d_t x_{st}) \\ &\qquad + 2c_2(d_s x_{st}, d_s d_t x_{st}) + 2c_2'(d_t p_{st}, d_s x_{st}, d_s d_t x_{st}) \\ &= c'(d_t p_{st}, d_s x_{st}) + \tfrac{1}{2}c''(d_t p_{st}, d_t p_{st}, d_s x_{st}) + c(a(d_s d_t w_{st})) + c(b(d_s x_{st}, d_t x_{st})) \\ &\qquad + c'(d_t p_{st}, b(d_s x_{st}, d_t x_{st})) + 2c_2(d_s x_{st}, b(d_s x_{st}, d_t x_{st})) \\ &\qquad + 2c_2'(d_t p_{st}, d_s x_{st}, b(d_s x_{st}, d_t x_{st})). \end{aligned}$$

Here we are writing c', c'' for the derivatives with respect to p. We set $\tilde{d} = d + n$ and combine this equation with the equation (5) to obtain a two-parameter equation for the $\mathbb{R}^{\tilde{d}}$-valued process $\tilde{x}_{st} = \begin{pmatrix} x_{st} \\ p_{st} \end{pmatrix}$, which we can write in the form

$$d_s d_t \tilde{x}_{st} = \tilde{a}(d_s d_t w_{st}) + \tilde{b}(d_s \tilde{x}_{st}, d_t \tilde{x}_{st}). \tag{41}$$

(The $\sim$ notation in this paragraph has nothing to do with that used in the paragraph on uniqueness above.) We impose regular semimartingale initial values $\tilde{x}_{s0} = \begin{pmatrix} x_{s0} \\ p_{s0} \end{pmatrix}$ and $\tilde{x}_{0t} = \begin{pmatrix} x_{0t} \\ p_{0t} \end{pmatrix}$, where $(p_{s0} : s \geqslant 0)$ is obtained by solving the one-parameter equation (40) along x_{s0}. Introduce the two companion equations for $\tilde{d} \times \tilde{d}$ matrix-valued processes $\tilde{u}_{st}$ and $\tilde{v}_{st}$

$$d_s \tilde{u}_{st} = \tilde{b}_{11}(d_s \tilde{x}_{st}, \cdot)\tilde{u}_{st} + \tilde{b}_{12}(d_s \tilde{x}_{st}, d_s \tilde{x}_{st}, \cdot)\tilde{u}_{st}, \tag{42}$$

$$d_t \tilde{v}_{st} = \tilde{b}_{11}(\cdot, d_t \tilde{x}_{st})\tilde{v}_{st} + \tilde{b}_{21}(\cdot, d_t \tilde{x}_{st}, d_t \tilde{x}_{st})\tilde{v}_{st}. \tag{43}$$

Impose boundary conditions for $\tilde{u}_{st}$ and $\tilde{v}_{st}$ analogous to those for u_{st} and v_{st}. Write (7) in the form

$$d_t \tilde{q}_{st} = \tilde{e}(d_t \tilde{x}_{st}). \tag{44}$$

By assumption, there exists a $K' < \infty$ which is both a uniform bound for a, b, c, e and is also a Lipschitz constant for b, c, c', c''. We can then find a uniform bound $\tilde{K} < \infty$ on $\tilde{a}, \tilde{b}, \tilde{e}$, which is also a Lipschitz constant for $\tilde{b}$, and which depends only on K'. The above argument shows that the system of equations (41–44) has a unique maximal regular solution $(\tilde{x}_{st}, \tilde{q}_{st}, \tilde{u}_{st}, \tilde{v}_{st} : (s,t) \in \tilde{\mathcal{D}})$, with the property that, as $(s,t) \uparrow \zeta(\tilde{\mathcal{D}})$, almost surely,

$$\tilde{m}_{st} := \sup_{r \leq s, u \leq t} |(\tilde{u}_{ru}, \tilde{u}_{ru}^{-1}, \tilde{v}_{ru}, \tilde{v}_{ru}^{-1})| \uparrow \infty.$$

Write

$$\tilde{x}_{st} = \begin{pmatrix} x^1_{st} \\ x^2_{st} \end{pmatrix}, \quad \tilde{u}_{st} = \begin{pmatrix} u^{11}_{st} & u^{12}_{st} \\ u^{21}_{st} & u^{22}_{st} \end{pmatrix}, \quad \tilde{v}_{st} = \begin{pmatrix} v^{11}_{st} & v^{12}_{st} \\ v^{21}_{st} & v^{22}_{st} \end{pmatrix},$$

and use analogous block notation for the tensors $\tilde{u}^*_{st}$ and $\tilde{v}^*_{st}$. Note that

$$\tilde{b}(d_s \tilde{x}_{st}, \cdot) = \begin{pmatrix} b(d_s x^1_{st}, \cdot) & 0 \\ f(d_s x^1_{st}) & c'(\cdot, d_s x^1_{st}) \end{pmatrix}, \quad \tilde{b}(\cdot, d_t \tilde{x}_{st}) = \begin{pmatrix} b(\cdot, d_t x^1_{st}) & 0 \\ g(d_t \tilde{x}_{st}) & 0 \end{pmatrix},$$

where

$$f(d_s x^1_{st}) = c(b(d_s x^1_{st}, \cdot)) + 2c_2(d_s x^1_{st}, b(d_s x^1_{st}, \cdot)),$$
$$g(d_t \tilde{x}_{st}) = c'(d_t x^2_{st}, \cdot) + \tfrac{1}{2}c''(d_t x^2_{st}, d_t x^2_{st}, \cdot) + c(b(\cdot, d_t x^1_{st})) + c'(d_t x^2_{st}, b(\cdot, d_t x^1_{st})).$$

Here, we have written $b(d_s x_{st}, \cdot)$ as a short form of $b_{11}(d_s x_{st}, \cdot) + b_{12}(d_s x_{st}, d_s x_{st}, \cdot)$, and analogously for $b(\cdot, d_t x_{st})$ and $\tilde{b}(d_s \tilde{x}_{st}, \cdot)$. On multiplying out in blocks, we see that the process $(x^1_{st}, x^2_{st}, \tilde{q}_{st}, u^{11}_{st}, (u^*_{st})^{111}, v^{11}_{st}, (v^*_{st})^{111} : (s,t) \in \tilde{\mathcal{D}})$ satisfies equations (5–11). Hence, we must have $\tilde{\mathcal{D}} \subseteq \mathcal{D}$ and $(x^1_{st}, x^2_{st}, \tilde{q}_{st}, u^{11}_{st}, v^{11}_{st}) = (x_{st}, p_{st}, q_{st}, u_{st}, v_{st})$ for all $(s,t) \in \tilde{\mathcal{D}}$. In particular, $(p_{st} : (s,t) \in \tilde{\mathcal{D}})$ is a semimartingale.

It remains to show that $\tilde{\mathcal{D}} = \mathcal{D}$, which we can do by showing that, almost surely, $\tilde{m}_{st}$ remains bounded on $\tilde{\mathcal{D}}_{M,N} = \tilde{\mathcal{D}} \cap \mathcal{D}_{M,N}$, for all $M, N \geqslant 1$. We first obtain a Hölder estimate in t for p_{st}. We have

$$d_s(p_{st} - p_{st'}) = c(p_{st}, d_s x_{st}) - c(p_{st'}, d_s x_{st'}),$$

where we have now made the dependence of c on p explicit. Set

$$f(s) = \mathbb{E}\left(|p_{st} - p_{st'}|^\alpha 1_{\{(s,t),(s,t') \in \tilde{\mathcal{D}}_{M,N}\}}\right).$$

We use the estimates (32) and (33) to obtain a constant $C = C(\alpha, K', M, N, T) < \infty$ such that

$$f(s) \leqslant C\left(|t - t'|^{\alpha/2} + \int_0^s f(r)dr\right).$$

This implies that $f(s) \leqslant C|t - t'|^{\alpha/2}$ for all $s \geqslant 0$ for a constant $C < \infty$ of the same dependence. We now know that, for such a constant $C < \infty$, we have

$$\mathbb{E}\left(|p_{s't'} - p_{st}|^\alpha 1_{\{(s,t),(s',t') \in \tilde{\mathcal{D}}_{M,N}\}}\right) \leqslant C(|s - s'|^{\alpha/2} + |t - t'|^{\alpha/2}). \tag{45}$$

We turn to $\tilde{u}_{st}$ and $\tilde{v}_{st}$. The following equations hold

$$d_s u^{12}_{st} = b(d_s x_{st}, \cdot) u^{12}_{st}, \quad d_t v^{12}_{st} = b(\cdot, d_t x_{st}) v^{12}_{st}, \quad d_t v^{22}_{st} = g(d_t \tilde{x}_{st}) v^{12}_{st}.$$

By uniqueness of solutions, we obtain $u^{12}_{st} = u_{st} u^{-1}_{0t} u^{12}_{0t}$ so, in particular, $u^{12}_{s0} = 0$. Similarly, $v^{12}_{st} = v_{st} v^{-1}_{s0} v^{12}_{s0}$, so $v^{12}_{0t} = 0$. Since $\tilde{u}_{0t} = \tilde{v}_{0t}$ and $\tilde{u}_{s0} = \tilde{v}_{s0}$, we deduce that $u^{12}_{st} = v^{12}_{st} = 0$. Then $d_t v^{22}_{st} = 0$, so $v^{22}_{st} = v^{22}_{s0} = u^{22}_{s0}$. We also have the equations

$$d_s u^{21}_{st} = f(d_s x_{st}) u_{st} + c'(., d_s x_{st}) u^{21}_{st}, \quad d_s u^{22}_{st} = c'(., d_s x_{st}) u^{22}_{st}, \quad d_t v^{21}_{st} = g(d_t \tilde{x}_{st}) v_{st}$$

and we note that

$$\tilde{u}^{-1}_{st} = \begin{pmatrix} u^{-1}_{st} & 0 \\ -(u^{22}_{st})^{-1} u^{21}_{st} u^{-1}_{st} & (u^{22}_{st})^{-1} \end{pmatrix}, \quad \tilde{v}^{-1}_{st} = \begin{pmatrix} v^{-1}_{st} & 0 \\ -(v^{22}_{st})^{-1} v^{21}_{st} v^{-1}_{st} & (v^{22}_{st})^{-1} \end{pmatrix},$$

and

$$d_s (u^{22}_{st})^{-1} = -(u^{22}_{st})^{-1} c'(., d_s x_{st}) + (u^{22}_{st})^{-1} c'(., d_s x_{st}) c'(., d_s x_{st}).$$

We use the inequalities (32), (33) and (45), and an easy variation of the argument leading to (34) and (38) to obtain a constant $C = C(\alpha, K', M, N, T) < \infty$ such that

$$\mathbb{E}\left(\left|(\tilde{u}_{s't'}, \tilde{u}^{-1}_{s't'}) - (\tilde{u}_{st}, \tilde{u}^{-1}_{st})\right|^\alpha 1_{\{(s,t),(s',t') \in \tilde{\mathcal{D}}_{M,N}\}}\right) \leqslant C(|s - s'|^{\alpha/2} + |t - t'|^{\alpha/2}). \tag{46}$$

Then, using [9, Theorem 3.2.1] as above, we can conclude that, almost surely, $(\tilde{u}_{st}, \tilde{u}^{-1}_{st})$ remains bounded on $\tilde{\mathcal{D}}_{M,N}$. It remains to show that the same is true for $(\tilde{v}_{st}, \tilde{v}^{-1}_{st})$ and, given the relations already noted, it will suffice to show this for v^{21}_{st}. We have

$$\begin{aligned} d_s \tilde{u}^*_{st} &= \tilde{u}^{-1}_{st} \{\tilde{b}_{12}(d_s \tilde{x}_{st}, \tilde{u}_{st} \cdot, \tilde{u}_{st} \cdot) \\ &\qquad + \tilde{b}_{22}(d_s \tilde{x}_{st}, d_s \tilde{x}_{st}, \tilde{u}_{st} \cdot, \tilde{u}_{st} \cdot) - \tilde{b}_{11}(d_s \tilde{x}_{st}, \tilde{b}_{12}(d_s \tilde{x}_{st}, \tilde{u}_{st} \cdot, \tilde{u}_{st} \cdot))\} \\ &= h(x_{st}, p_{st}, \tilde{u}_{st}, \tilde{u}^{-1}_{st}, d_s x_{st}), \end{aligned}$$

where h is defined by the final equality and where we have used (6) to write $d_s \tilde{x}_{st}$ in terms of $d_s x_{st}$. A variation of the argument used for $\tilde{u}_{st}$ shows that, almost surely, $\tilde{u}^*_{st}$ remains bounded on $\tilde{\mathcal{D}}_{M,N}$. Then, we can use the $\sim$ and t-analogue of equations (29) and (30) to express v^{21}_{st} as a sum of integrals with respect to $(x_{0t}, p_{0t} : t \geqslant 0)$ and $(w_{st} : s, t \geqslant 0)$. This

leads, as above, to L^α-Hölder estimates which allow us to conclude that, almost surely, v^{21}_{st} remains bounded on $\tilde{\mathcal{D}}_{M,N}$, as required. $\square$

5. Derivation of the formula

Let $(w_{st} : s, t \geqslant 0)$ be an $\mathbb{R}^m$-valued Brownian sheet and let $(z_{s0} : s \geqslant 0)$ be an independent $\mathbb{R}^m$-valued Brownian motion. Thus $w_{st} = (w^1_{st}, \dots, w^m_{st})$ and $z_{s0} = (z^1_{s0}, \dots, z^m_{s0})$, and each component process is an independent scalar Brownian sheet, or Brownian motion, respectively. The two-parameter hyperbolic stochastic differential equation

$$d_s d_t z_{st} = d_s d_t w_{st} - \tfrac{1}{2} d_s z_{st} dt, \quad s, t \geqslant 0, \tag{47}$$

with given boundary values $(z_{s0} : s \geqslant 0)$ and $z_{0t} = 0$, for $t \geqslant 0$, has a unique solution $(z_{st} : s, t \geqslant 0)$. Set $z_t = (z_{st} : s \geqslant 0)$, then $(z_t)_{t \geqslant 0}$ is a realization of the Ornstein-Uhlenbeck process on the m-dimensional Wiener space. See [8] or [9]. The Stratonovich form of (47) is given by

$$\partial_s \partial_t z_{st} = \partial_s \partial_t w_{st} - \tfrac{1}{2} \partial_s z_{st} \partial t, \quad s, t \geqslant 0.$$

Fix $x \in \mathbb{R}^d$ and consider for each $t \geqslant 0$ the Stratonovich stochastic differential equation

$$\partial_s x_{st} = X_i(x_{st}) \partial_s z^i_{st} + X_0(x_{st}) \partial s, \quad s \geqslant 0,$$

with initial value $x_{0t} = x$. This can be written in Itô form as

$$d_s x_{st} = X_i(x_{st}) d_s z^i_{st} + \tilde{X}_0(x_{st}) ds, \quad s \geqslant 0, \tag{48}$$

where $\tilde{X}_0 = X_0 + \frac{1}{2} \sum_{i=1}^d \nabla X_i . X_i$. Consider also, for each $t \geqslant 0$, the stochastic differential equation

$$\partial_s U_{st} = \nabla X_i(x_{st}) U_{st} \partial_s z^i_{st} + \nabla X_0(x_{st}) U_{st} \partial s, \quad s \geqslant 0,$$

with initial value $U_{0t} = I$, and its Itô form

$$d_s U_{st} = \nabla X_i(x_{st}) U_{st} d_s z^i_{st} + \nabla \tilde{X}_0(x_{st}) U_{st} ds, \quad s \geqslant 0. \tag{49}$$

Proposition 5.1. *There exist (two-parameter) semimartingales $(z_{st} : s, t \geqslant 0)$, $(x_{st} : s, t \geqslant 0)$ and $(U_{st} : s, t \geqslant 0)$ such that $(z_{st} : s, t \geqslant 0)$ satisfies (47) and, for all $t \geqslant 0$, $(x_{st} : s \geqslant 0)$ and $(U_{st} : s \geqslant 0)$ satisfy (48) and (49), with the boundary conditions given above. Moreover, almost surely, U_{st} is invertible for all $s, t \geqslant 0$.*

Proof. We seek to apply Theorem 4.2. There are three minor obstacles: firstly to deal with the ds and dt differentials appearing in the equations, secondly, to show that the domain of the solutions is the whole of $(\mathbb{R}^+)^2$ and, thirdly, to deal with the fact that the coefficients in (49) do not have the required boundedness of derivatives.

Let us introduce a further equation

$$d_s d_t z^0_{st} = 0,$$

with boundary conditions $z^0_{s0} = s$ and $z_{0t} = t$ for all $s, t \geqslant 0$. We then replace dt and ds in (47) and (48), respectively, by $d_t z^0_{st}$ and $d_s z^0_{st}$. When we obtain a solution, it will follow that $z^0_{st} = s + t$, so $d_t z^0_{st} = dt$ and $d_s z^0_{st} = ds$, as required.

In order to show that $\mathcal{D} = (\mathbb{R}^+)^2$, it will suffice to show that the companion processes u_{st} and v_{st} associated with the equations

$$d_s d_t z^0_{st} = 0, \quad d_s d_t z_{st} = d_s d_t w_{st} - \frac{1}{2} d_s z_{st} d_t z^0_{st},$$

according to equations (8) and (10), along with their inverses, remain bounded on compacts in s and t. We leave this to the reader.

Finally, choose for each $M \in \mathbb{N}$ a smooth and compactly supported function ψ_M on $\mathbb{R}^d \otimes (\mathbb{R}^d)^*$, such that $\psi_M(U) = U$ whenever $|U| \leqslant M$. We can apply Theorem 4.2 to the system (47), (48), together with the modified equation

$$d_s U^M_{st} = \nabla X_i(x_{st})\psi_M(U^M_{st})d_s z^i_{st} + \nabla \tilde{X}_0(x_{st})\psi_M(U^M_{st})ds.$$

Define

$$\mathcal{D}_M = \{(s,t) : |U^M_{s't'}| \leqslant M \text{ for all } s' \leqslant s, t' \leqslant t\}.$$

By local uniqueness, we can define consistently U on $\mathcal{D} = \cup_M \mathcal{D}_M$ by $U_{st} = U^M_{st}$ for $(s,t) \in \mathcal{D}_M$. By some straightforward estimation using the one-parameter equations (49), we obtain, for all $T < \infty$ and all $p \in [1,\infty)$, a constant $C < \infty$ such that

$$\sup_{s,s',t,t' \leqslant T} \mathbb{E}(|U_{st} - U_{s't'}|^p 1_{\{(s,t),(s't') \in \mathcal{D}\}}) \leqslant C(|s-s'|^{p/2} + |t-t'|^{p/2}).$$

Then, by [9, Theorem 3.2.1], almost surely, U is bounded uniformly on $\mathcal{D} \cap [0,T]^2$. Hence $\mathcal{D} = (\mathbb{R}^+)^2$, and we have obtained the desired semimartingale U. The invertibility of U can be proved by applying the same argument to the usual equation for the inverse. □

By the Stratonovich chain rule,

$$\partial_s\partial_t x_{st} = \nabla X_i(x_{st})\partial_s z^i_{st}\partial_t x_{st} + \nabla X_0(x_{st})\partial s\partial_t x_{st} + X_i(x_{st})\partial_s\partial_t z^i_{st}.$$

Now

$$\begin{aligned}\partial_s\partial_t U_{st} = {}& \nabla X_i(x_{st})\partial_s z^i_{st}\partial_t U_{st} + \nabla X_0(x_{st})\partial s\partial_t U_{st} \\ & + (\nabla^2 X_i(x_{st})\partial_t x_{st})U_{st}\partial_s z^i_{st} + (\nabla^2 X_0(x_{st})\partial_t x_{st})U_{st}\partial s + \nabla X_i(x_{st})U_{st}\partial_s\partial_t z^i_{st},\end{aligned}$$

so

$$\partial_t U_{st}\partial_s\partial_t z^i_{st} = \tfrac{1}{2}\partial_s\partial_t U_{st}\partial_s\partial_t w^i_{st} = \tfrac{1}{2}\nabla X_i(x_{st})U_{st}\partial s\partial t$$

and

$$\partial_s(U^{-1}_{st}\partial_t U_{st}) = U^{-1}_{st}\left\{\nabla^2 X_i(x_{st})\partial_s z^i_{st}\partial_t x_{st} + \nabla^2 X_0(x_{st})\partial s\partial_t x_{st} + \nabla X_i(x_{st})\partial_s\partial_t z^i_{st}\right\}U_{st}.$$

Define also a two-parameter, $\mathbb{R}^d$-valued, semimartingale $(y_{st} : s,t \geqslant 0)$ by

$$\partial_t y_{st} = U^{-1}_{st}\partial_t x_{st}, \quad y_{s0} = 0.$$

Then

$$\partial_s\partial_t y_{st} = U^{-1}_{st}X_i(x_{st})\partial_s\partial_t z^i_{st}.$$

Note that

$$\partial_t y_{st}\partial_s\partial_t z^i_{st} = \partial_t y_{st}\partial_s\partial_t w^i_{st} = \tfrac{1}{2}\partial_s\partial_t y_{st}\partial_s\partial_t w^i_{st} = \tfrac{1}{2}U^{-1}_{st}X_i(x_{st})\partial s\partial t.$$

So

$$\partial_s(\partial_t y_{st} \otimes \partial_t y_{st}) = \partial_s\partial_t y_{st} \otimes \partial_t y_{st} + \partial_t y_{st} \otimes \partial_s\partial_t y_{st} = U^{-1}_{st}X_i(x_{st}) \otimes U^{-1}_{st}X_i(x_{st})\partial s\partial t.$$

Note also that

$$\partial_s(U^{-1}_{st}X_i(x_{st})) = U^{-1}_{st}[X_i, X_j](x_{st})\partial_s z^j_{st} + U^{-1}_{st}[X_i, X_0](x_{st})\partial s.$$

So

$$\partial_s(U^{-1}_{st}X_i(x_{st}))\partial_s\partial_t z^i_{st} = U^{-1}_{st}[X_i, X_j](x_{st})\partial_s z^j_{st}(\partial_s\partial_t w^i_{st} - \tfrac{1}{2}\partial_s z^i_{st}\partial t) = 0.$$

Moreover

$$\partial_t(U_{st}^{-1}X_i(x_{st}))d_s\partial_t z_{st}^i = \partial_t(U_{st}^{-1}X_i(x_{st}))d_s\partial_t w_{st}^i = 0.$$

Hence, we have

$$d_s d_t y_{st} = U_{st}^{-1}X_i(x_{st})d_s d_t z_{st}^i = U_{st}^{-1}X_i(x_{st})(\partial_s\partial_t w_{st}^i - \tfrac{1}{2}\partial_s z_{st}^i \partial t).$$

We compute

$$\begin{aligned}
&\partial_s(U_{st}^{-1}\partial_t U_{st}\partial_t y_{st}) \\
&\qquad = U_{st}^{-1}\left\{\nabla^2 X_i(x_{st})\partial_s z_{st}^i + \nabla^2 X_0(x_{st})\partial s\right\}\partial_t x_{st}\otimes\partial_t x_{st} + U_{st}^{-1}\nabla X_i(x_{st})X_i(x_{st})\partial s\partial t.
\end{aligned}$$

Define

$$R_{st} = -\int_0^s U_{rt}^{-1}X_i(x_{rt})d_r z_{rt}^i, \quad C_{st} = \int_0^s U_{rt}^{-1}X_i(x_{rt})\otimes U_{rt}^{-1}X_i(x_{rt})dr.$$

Our calculations show that the $(\mathcal{F}_{st} : t \geqslant 0)$-semimartingale $(y_{st} : t \geqslant 0)$ has finite-variation part $(\bar{y}_{st} : t \geqslant 0)$ and quadratic variation given by

$$d_t\bar{y}_{st} = \tfrac{1}{2}R_{st}dt, \quad \partial_t y_{st}\otimes\partial_t y_{st} = C_{st}dt.$$

Moreover

$$d_t x_{st} = U_{st}d_t y_{st} + \tfrac{1}{2}\partial_t U_{st}\partial_t y_{st},$$

so $(x_{st} : t \geqslant 0)$ has finite-variation part $(\bar{x}_{st} : t \geqslant 0)$ and quadratic variation given by

$$d_t\bar{x}_{st} = \tfrac{1}{2}L_{st}dt, \quad \partial_t x_{st}\otimes\partial_t x_{st} = \Gamma_{st}dt,$$

where

$$\begin{aligned}
L_{st} = U_{st}R_{st} + U_{st}&\int_0^s U_{rt}^{-1}\{\nabla^2 X_i(x_{rt})\partial_r z_{rt}^i + \nabla^2 X_0(x_{rt})\partial r\}\Gamma_{rt} \\
&+ U_{st}\int_0^s U_{rt}^{-1}\nabla X_i(x_{rt})X_i(x_{rt})\partial r
\end{aligned}$$

and where $\Gamma_{st} = U_{st}C_{st}U_{st}^*$.

Note that both $(\Gamma_{st} : t \geqslant 0)$ and $(L_{st} : t \geqslant 0)$ are stationary processes and that, by standard one-parameter estimates, Γ_{s0} and L_{s0} have finite moments of all orders. By Itô's formula, for any C^2 function f, setting $f_{st} = f(x_{st})$, the process $(f_{st} : t \geqslant 0)$ is an $(\mathcal{F}_{st} : t \geqslant 0)$-semimartingale with finite-variation part $(\bar{f}_{st} : t \geqslant 0)$ and quadratic variation given by

$$d_t\bar{f}_{st} = \tfrac{1}{2}\left(L_{st}^i\nabla_i f(x_{st}) + \Gamma_{st}^{ij}\nabla_i\nabla_j f(x_{st})\right)dt, \quad \partial_t f_{st}\partial_t f_{st} = \nabla_i f(x_{st})\Gamma_{st}^{ij}\nabla_j f(x_{st})dt.$$

In particular, if $m_{st} = f_{st} - f_{s0} - \bar{f}_{st}$, then $(m_{st} : t \geqslant 0)$ is a (true) martingale. Hence, for $f, g \in C_b^2(\mathbb{R}^d)$, we obtain the integration-by-parts formula

$$\begin{aligned}
&\mathbb{E}[\nabla_i f(x_{s0})\Gamma_{s0}^{ij}\nabla_j g(x_{s0})] = \lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}\left[\{f(x_{st}) - f(x_{s0})\}\{g(x_{st}) - g(x_{s0})\}\right] \\
&\quad = -2\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}[f(x_{s0})\{g(x_{st}) - g(x_{s0})\}] = -\mathbb{E}[f(x_{s0})\{L_{s0}^i\nabla_i g(x_{s0}) + \Gamma_{s0}^{ij}\nabla_i\nabla_j g(x_{s0})\}].
\end{aligned}$$

An obvious limit argument allows us to deduce the following simple formula, corresponding to the case $g(x) = x^j$. For all $f \in C_b^2(\mathbb{R}^d)$ and for $j = 1, \ldots, d$, we have

$$\mathbb{E}[\nabla_i f(x_{s0})\Gamma_{s0}^{ij}] = -\mathbb{E}[f(x_{s0})L_{s0}^j].$$

The general formula can then be recovered by replacing f by $f\nabla_j g$ and summing over j.

The basic observation underlying this formula is that the distributions of (z_0, z_t) and (z_t, z_0) are identical, and hence that the same is true for (x_{s0}, x_{st}) and (x_{st}, x_{s0}), when $(x_{st} : s \geqslant 0)$ is obtained by solving a stochastic differential equation driven by $(z_{st} : s \geqslant 0)$, with initial condition independent of t. In fact a stronger notion of reversibility is true. The distributions of $(z_{su} : s \geqslant 0, u \in [0,t])$ and $(z_{s,t-u} : s \geqslant 0, u \in [0,t])$ are identical, and hence the same is true for $(x_{su} : s \geqslant 0, u \in [0,t])$ and $(x_{s,t-u} : s \geqslant 0, u \in [0,t])$. This may be combined with the fact that the Stratonovich integral is invariant under time-reversal to see that

$$\mathbb{E}\left[\{f(x_{st}) - f(x_{s0})\} \int_0^t U_{su}^{-1} \partial_u x_{su}\right] = -2\mathbb{E}\left[f(x_{s0}) \int_0^t U_{su}^{-1} \partial_u x_{su}\right].$$

From this identity, by a similar argument, we obtain the following alternative integration-by-parts formula. For all $f \in C_b^2(\mathbb{R}^d)$, we have

$$\mathbb{E}[\nabla f(x_{s0}) U_{s0} C_{s0}] = -\mathbb{E}[f(x_{s0}) R_{s0}].$$

This formula is the variant discovered by Bismut, which is closely related to the Clark–Haussmann formula.

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